CS 6200: Algorithmics II	**Fall 2020**
Instructor: Avah Banerjee, S&T.	
Lectures: lecture 6	**Dates: 09/04/2020**

1 Matching matroid

Definition 1.1 *For a given graph $G(V, E)$, V is vertices set, E is edge set, there is a matroid $M(V, I)$: $U \in I$ if there is a matching that covers at-least vertices in U.*
V: ground set. I: independent set.

To prove the above statement, we only need to prove that the $M(V, I)$ has the following three properties of Matroid

1. $\emptyset \in I$
2. **hereditary property**: $A \in I, B \subseteq A \quad => \quad B \in I$
3. **augmentation property**: $|A| > |B| \ and \ A, B \in I \quad \exists\, e \in (A - B)\ satisfy\ B \cup \{e\} \in I$

Proof: of the above properties.

1. It is obviously true for $M(V, I)$
2. If there is a matching M_A that covers A, then $\forall\ B \subseteq A$, there is obvious a $M_B \subseteq M_A$ that cover B.
3. Because of $|A| > |B|$, there must be a pair of vertices $b_1, b_2 \in B$ matched an edge $E_B \in M_B$ and also matched two edges $E_{A1}, E_{A2} \in M_A$ with $E_{A1} = (a_1, b_1), E_{A2} = (a_2, b_2)$. In this situation, we could remove E_B from M_B and add E_{A1}, E_{A2} to M_B. then the 3 requirement meet.

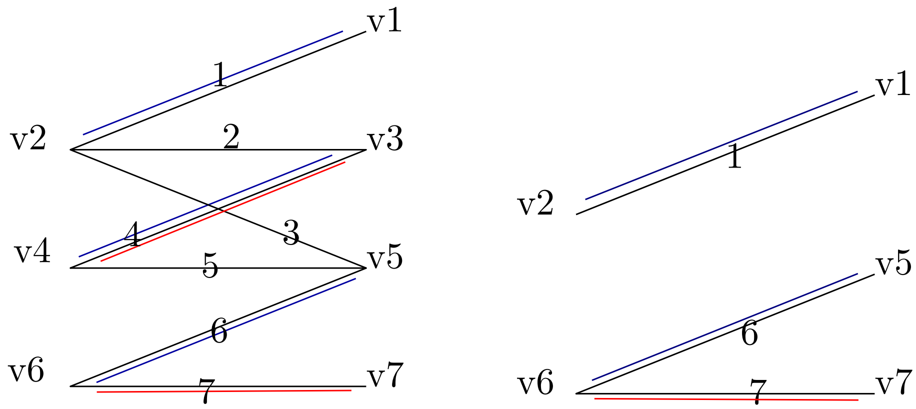
■

Detailed proof of *augmentation property* Assume $A \in I, B \in I$, $A = \{v_1, v_2, v_4, v_5, v_6\}$ and $B = \{v_3, v_4, v_6, v_7\}$ (Fig. 1.1(a)). M_A and M_B are the matchings of A and B , $M_A = \{1, 4, 6\}$ (blue edges in Fig. 1.1(a)) $M_B = \{4, 7\}$(red edges in Fig. 1.1(a)) . We could get

$$M' = (M_A - M_B) \cup (M_B - M_A) = \{1, 6, 7\}$$

$G(M')$(Fig. 1.1(b)) is the induced subgraph corresponding to the edges in the M'. And we color the edges in M_A with blue and edges in M_B with red. Each edge in M' is either red or blue.

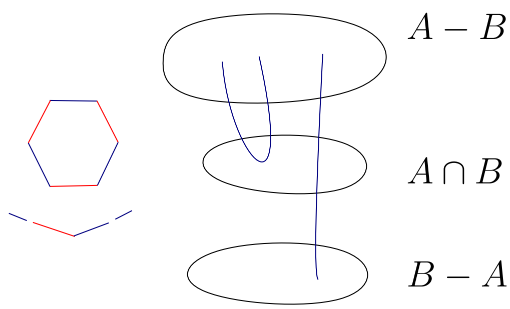
$$|A| > |B| \Rightarrow |A - B| > |B - A|$$



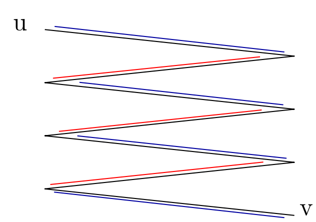
(a) matching (b) $G(M')$

Fig. 1.1

We could find that each vertex in $A - B$ start an alternating path (blue ... red ... blue ....)(Fig. 1.2(a)) So, in the case above, we could substitute all the edges in M_B with the edges in M_A(Fig. 1.2(b)), then all the vertexes still be covered by a matching which is larger than the original one.



(a) A B details



(b) transform

Fig. 1.2

2 Weighted matroid

Definition 2.1 *$M(E, I)$ is a matroid, E is ground set. I is independent set.*
weight mapping on E $\quad w : E \to R^+$

From the definition, we could find that weighted matroid is just a normal matroid that equipped with a weight mapping on ground set elements to real number. This new feature could help us find the maximal value on Independent sets. An independent set's weight value can be calculated by following formula.

$$A \subseteq I, w(A) = \sum_{e \in A} w(e)$$

weighted matroid is a very useful tool to solve problems. The following are two examples.

Example 2.1 *Minimum spanning forest.*
$G(V, E)$ is a forest and $M(V, I)$ is a matroid of this forest graph. Of this matroid,
Independent set = forest
Maximal Independent set = spanning forests.
$w'(e) = max\, w(e) - w(e)$
$w'(A) = \sum_{e \in A} w'(e) = |A|.w_0 - w(A)$

Example 2.2 *Finding the rank of a matrix*
A is a matrix over F
$\mathbb{I} =$ *collection of set of independent column vectors.*
E = set of all the column vector of A.
$w(e) = 1$
$w(I) = |I| \ \ , \ \ I \in \mathbb{I}$

Weighted matroid has a very good property that the maximal weight value of independent sets can be calculated with greedy algorithm. Usually, greedy algorithm is very efficient. The following is that greedy algorithm.

Algorithm 1 Greedy algorithm

1: **Input:** A Weighted Matroid $M(E, I)$ with empty I.
2: **Output:** A maximal independent set I.
3: Sort E with value $w(e)$ in the monotonically decreasing order
4: $I = \emptyset$.
5: For $e \in E$ (based on the sorted order)
   if $I \cup \{e\} \in \mathbb{I}$ then $I \cup \{e\} \to I$
6: **return** I.