CS 6200: Algorithmics II	**Fall 2020**
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Lectures: 5	**Dates: 9/2**

1 Matroid

A matroid is a structure that abstracts and generalizes the notion of linear independence in vector spaces, which was introduced in 1935 by Whitney and Nakasawa independently. There are many different equivalent ways to define a matroid.

Definition 1.1 *Suppose there is a set of vectors* $\{\vec{v}_1, ..., \vec{v}_k\}$ *over a field* $\mathcal{F}$*. If there exist certain scalars* $\lambda_1, ..., \lambda_k$ *in the same field* $\mathcal{F}$*, and they are not all zero, such that* $\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2 +, ..., \lambda_k\vec{v}_k = \vec{0}$*, where* $\vec{0}$ *denotes the zero vector. Then the set of vectors* $\{\vec{v}_1, ..., \vec{v}_k\}$ *is said to be linearly dependent.*

Example 1.1 *In the GF(2), it is know that: 0+0=0, 0+1=1, 1+1=0. Suppose A is the following matrix in the GF(2).*

$$A = \begin{array}{c} \\ v1 \\ v2 \\ v3 \\ v4 \\ v5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e1 & e2 & e3 & e4 & e5 & e6 & e7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

In the matrix A, we have:

$$e_2 + e_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = e_5$$

Since $\{e_2, e_4\}, \{e_4, e_5\}, \{e_2, e_5\}$*are all in independent,* $\{e_2, e_4, e_5\}$ *is not only a dependent set, also a minimal dependent set.*

In the matrix A, let E be the set of all column vectors: $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, and let $\mathcal{C}$ be the minimal dependent sets, which are called circuits: $\mathcal{C} = \{\{e_7\}, \{e_2, e_3\}, \{e_2, e_4, e_5\}, \{e_3, e_4, e_5\}\}$. The pair $(E, \mathcal{C})$ is a matroid.

With the above example, the definition of a matroid using circuits can be obtained as following.

Definition 1.2 *A matroid M is a pair* $(E, \mathcal{C})$*, where E is a finite set, called the ground set, and* $\mathcal{C}$ *is the collection of subsets of E, called circuits, which satisfies:*
(1) $\varnothing \notin \mathcal{C}$*.*

(2) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are in $\mathcal{C}$, and $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\mathcal{C}_1 = \mathcal{C}_2$.
(3) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are in $\mathcal{C}$, $\mathcal{C}_1 \neq \mathcal{C}_2$, and $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \varnothing$, then $\forall\ e \in \mathcal{C}_1 \cap \mathcal{C}_2$, and $\mathcal{C}$ contains a member $\mathcal{C}_3$, such that $\mathcal{C}_3 \subseteq (\mathcal{C}_1 \cup \mathcal{C}_2) - \{e\}$.

The pair $(E, \mathcal{C})$ is also called the vector matroid of the matrix A and is denoted by $M[A]$.

Example 1.2 *Suppose there is a graph G in Figure 1, the matrix A above is the incidence matrix of the graph G. All spanning trees of the G are shown in Figure 2. These spanning trees are the maximal independent sets of the graph G.*

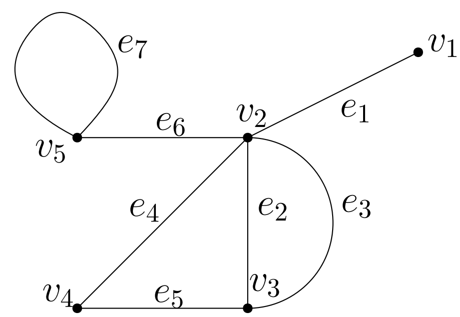

Fig. 1: The graph G in example 1.2

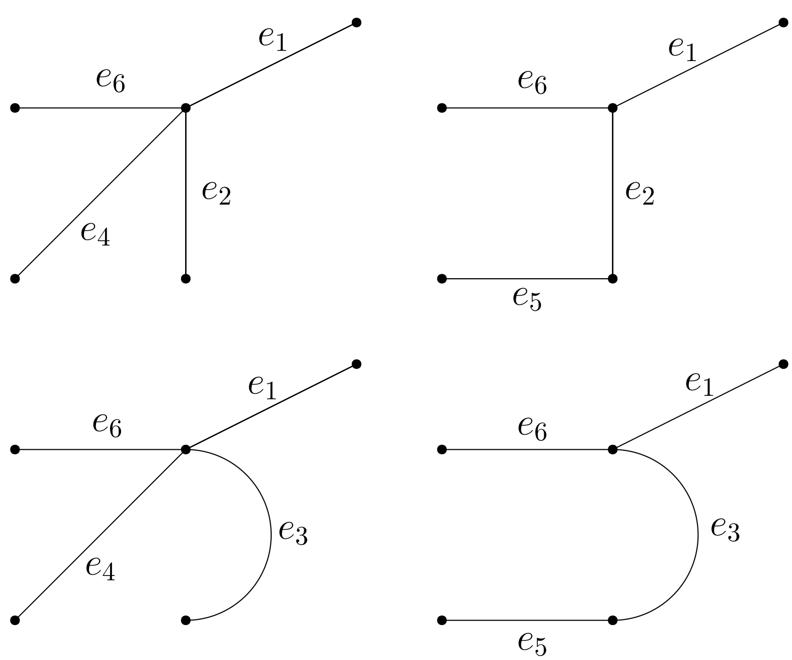

Fig. 2: All spanning trees of the graph G in example 1.2

Definition 1.3 *A set $\mathcal{I} \subseteq E$ is independent if there are no circuits contained in $\mathcal{I}$. If $\mathcal{I}$ is the maximum among the set of independent sets, then it is called maximal independent set.*

In Figure 2, each spanning tree is maximal independent. A maximal independent set is an independent set that is not a subset of any other independent set. In terms of independence, a matroid can be defined as following.

Definition 1.4 *A matroid M is a pair $(E, \mathcal{I})$, where E is a finite set, called the ground set, and $\mathcal{I}$ is a collection of subsets of E, which satisfies the following properties:*
(1) The empty set is independent, i.e.,$\mathcal{I} \neq \varnothing$.
(2) Every subset of an independent set is independent, i.e.,If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$. This is called hereditary property
(3) If A and B are in $\mathcal{I}$ and $\mid A \mid<\mid B \mid$, then there exists $e \in B - A$, such that $A \cup \{e\} \in \mathcal{I}$. This is called augmentation property

With the definition 1.4 (3), suppose A and B are maximal independent sets, we have $|A| = |B|$. All these maximal independent sets have the same sizes, and they are the *bases* of the matroid M. A basis of a matroid is a maximal independent set. Analogously, a circuit is a minimally dependent set of a matroid. The collection of maximal independent sets are know as the set of bases $\mathcal{B}$. The *rank* of the matroid M is the size of the sets in the bases $\mathcal{B}$.

Example 1.3 *Let $G = (V, E)$ be a graph, the matching matroid $M = (V, \mathcal{I})$ for G corresponds to $U \subseteq V$ is independent if there exists a matching that covers all of U (and possibly other vertices). A matching is a set of vertex disjoint edges. In Figure 3 (a), $\{e_1, e_5\}$ is a matching, however, $\{e_1, e_4\}$ is not a matching. In Figure 3 (b), if $U = \{v_1, v_2, v_3\}$, there is a matching $\{\{v_1, v_2\}, \{v_3, v_4\}\}$ that covers vertices in U.*

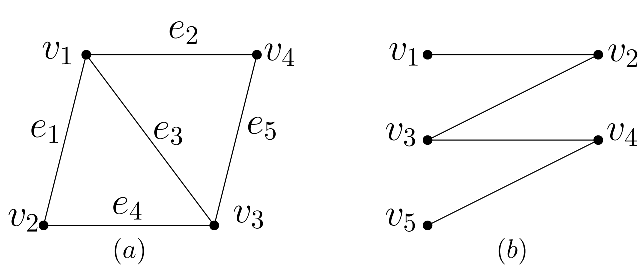

Fig. 3: Graphs in example 1.3

References and Further Reading

[1] Cormen, T.H., Leiserson, C.E., Rivest, R.L. & Stein, C., (2009). Introduction to algorithms. MIT press. [Chapter 16-4].

[2] Oxley, J., (2014). BRIEFLY, WHAT IS A MATROID?.