CS 6200: Algorithmics II	**Fall 2020**
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Lectures: 20	**Dates: 10/12**

1 One property of the LP's solution.

Our analysis is based on the standard form of LP problem.

$$\begin{aligned} min \quad & c^T x \\ s.t. \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Although LP problem can be expressed in other forms, but with some transformations, they all can be expressed as the above form.

Theorem 1.1 *If LP has an optimal Solution, then one of the solutions is at a vertex of $\mathcal{P}$. $\mathcal{P}$ is the feasible area.*

Proof: Suppose Otherwise. then

$$\begin{aligned} \exists d, \quad S.t. \quad |d| > 0 : \qquad & x \pm d \in \mathcal{P} \\ A\left(x \pm d\right) &= b \\ Ax &= b \\ \Rightarrow \qquad Ad &= 0 \end{aligned}$$

From the above steps we get conclusion that Ad=0, that means Ax=b holds True, and x is on the vertex of $\mathcal{P}$ ■

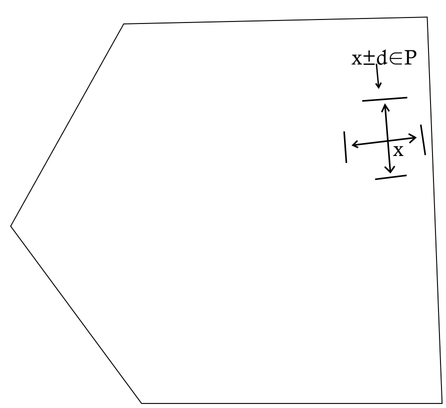

Fig. 1: $x \pm d \in \mathcal{P}$

We can analyze this conclusion from two aspects. We assume $c^T d \geq 0$.
Case-1 $(\exists\ j\ ,\ d_j < 0) \Rightarrow \exists \lambda > 0$ *and* $x' = x + \lambda d$ $s.t.$ $\exists k$ $\quad {x'}_k = 0 \quad , \quad x_k > 0$

$$c^T x' = c^T \left(x + \lambda d\right) = c^T x + \lambda c^T d \geq c^T x$$

The above formulas tell us that we can move x on the edge of the $\mathcal{P}$, if we reach a vertex, then we can enlarge the optimal value.
Case-2 $(\forall j\ ,\ d_j \geq 0)\, if\ \ c^T d > 0 \Rightarrow when\ \lambda \to \infty, c^T(x+\lambda d) \to \infty$. In this situation, only $c \perp d$ or $c^T d = 0$ then we can have meaningful $min \quad c^T x$. Also, we can try another d that satisfy the Case-1.

2 Basic Feasible Solution

$$\mathcal{P} = x | Ax = b, \quad x \geq 0$$

$\mathcal{P}$ is called feasible region. x is an $m+n$ dimension vector, it not only include original variables but also slack variables. A is $m \times (m+n)$ dimension matrix.
Let $B = i_1, i_2, \ldots, i_m$ B is the subset of A. If B is non-singular or is full rank, then a solution for $Ax = b$ can be expressed as below. x_B is the corresponding variable

$$A_B x_B = b$$

If $|B| < m$ then should choose some different columns from A.
We call $x_B = A_B^{-1} b$ a basic feasible solution if $x_B \geq 0$. Because $x \geq 0$. There is the possibility that some elements of x_B are not greater than 0, then it is not a feasible solution. Let's make a simple example
We have the following inequations.

$$\begin{cases} e_1: & 2x_1 - x_2 \leq 6 \\ e_2: & x_1 + x_2 \leq 6 \\ e_3: & x_2 \leq 3 \\ e_4: & -x_1 + x_2 \leq 2 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{cases} \rightarrow \begin{cases} e_1\prime: & 2x_1 - x_2 + s_1 = 6 & s_1 \geq 0 \\ e_2\prime: & x_1 + x_2 + s_2 = 6 & s_2 \geq 0 \\ e_3\prime: & x_2 + s_3 = 3 & s_3 \geq 0 \\ e_4\prime: & -x_1 + x_2 + s_4 \leq 2 & s_4 \geq 0 \end{cases} \rightarrow$$

$$A = \begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & s_3 & s_4 \end{matrix} \\ \begin{bmatrix} 2 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{matrix} \quad b = \begin{bmatrix} 6 \\ 6 \\ 3 \\ 2 \end{bmatrix}$$

We choose $B = \{i_1, i_2, i_3, i_5\} = \begin{bmatrix} 2 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix}$ we get $A_B^{-1} = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \\ 1 & -1/2 & 0 & 3/2 \\ 0 & -1/2 & 1 & -1/2 \end{bmatrix}$

$$\rightarrow x_B = A_B^{-1}b = \begin{bmatrix} 2 \\ 4 \\ 6 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_3 \end{bmatrix} \quad \rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

The above x is a basic solution, but not a feasible one, because $s_3 = -1$ violate the constrain.

So, we choose another $B = \{i_3, i_4, i_5, i_6\} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow A_B^{-1} = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \\ 1 & -1/2 & 0 & 3/2 \\ 0 & -1/2 & 1 & -1/2 \end{bmatrix} \rightarrow$

$A_B^{-1}b = \begin{bmatrix} 6 \\ 6 \\ 3 \\ 2 \end{bmatrix} \rightarrow x = \begin{bmatrix} 0 \\ 0 \\ 6 \\ 6 \\ 3 \\ 2 \end{bmatrix}$ this is a feasible one.

3 Pivoting

Pivoting is a technic used in Simplex method to find a better optimal value. It solves the problem in this way. First, we find a feasible solution for the LP problem.

Second, find entering variable. We express the target function by the linear combination of non-basis variables. Check the coefficient of these non-basis variables, and find the variable x_g who has the greatest positive coefficient among them. That means this variable has the fastest increasing rate. Third, find a leaving basic variable. We find it by the minimal ratio rule. The smallest one x_g/x_i x_i is the basis variables. Four, exchange the entering variable with leaving variable, repeat the steps from One to Third until we find that all the of optimal value expression's coefficient are all negative. That means the optimal value is not able to be increased.

Let's take an example.

$$\begin{aligned} &max \quad z = 4x_1 + 3x_2 \\ &\begin{cases} 2x_1 + 3x_2 + x_1 = 24 \\ 3x_1 + 2x_2 + x_4 = 26 \end{cases} \rightarrow A = \begin{matrix} x_1 & x_2 & x_3 & x_4 \\ \end{matrix} \begin{bmatrix} 2 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 24 \\ 26 \end{bmatrix} \end{aligned}$$

We find an initial basis B from A.

$B = \{i_3, i_4\}$, and the non-basis is $N = \{i_1, i_2\}$

We already have a feasible solution for this LP problem.

Then let's find the entering variable. For $max \quad z = 4x_1 + 3x_2$, so, x_1 is the entering variable.

Then let's find the leaving variable.

The ratio for x_3 is $24/2 = 12$

The ratio for x_4 is $26/3$

x_4 has the minimal ratio. So, it is the leaving variable. Then we use the new $B = \{i_1, i_3\}$ as the basis. And we get the following result

$max \quad z = \frac{1}{3}x_2 - \frac{4}{3}x_4$

$\begin{cases} \frac{5}{3}x_2 + x_3 - \frac{2}{3}x_4 = \frac{20}{3} \\ x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_4 = \frac{26}{3} \end{cases}$

We found there is still a positive coefficient in z. So, we need to continue. The entering variable is obvious x_2.

The ratio for x_3 is $\frac{\frac{20}{3}}{\frac{5}{3}} = 4$

The ratio for x_1 is $\frac{\frac{26}{3}}{\frac{2}{3}} = 13$

So, x_3 should be the leaving variable. Then we continue the calculation. And we use the new $B = \{i_1, i_2\}$ as the basis. We get the following result $\begin{matrix} max \quad z = -\frac{1}{5}x_3 - \frac{6}{5}x_4 \\ \begin{cases} x_2 + \frac{3}{5}x_3 - \frac{2}{5}x_4 = 4 \\ x_1 - \frac{2}{5}x_3 + \frac{3}{5}x_4 = 6 \end{cases} \end{matrix}$ We find that all the coefficient of z have already been negative. So, it reaches the optimal.The optimal solution is $x = x_1, x_2, x_3, x_4 = 6, 4, 0, 0$ and $max \quad z = 36$