CS 6200: Algorithmics II	**Fall 2020**
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1 Generic Push Relabel Algorithm(GPR)

The generic push-relabel algorithm uses the subroutine *INITIALIZE-PREFLOW* T to create an initial preflow in the flow network.

Algorithm [1] creates an initial preflow f defined by,

$$(u.v).f = \begin{cases} c(u,v) & \text{if u = s} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Then the height function is defined by,

$$h(u) = \begin{cases} n & \text{if u = s} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(2) defines the height function because the only edges (u, v) for which $u.h > v.h + 1$ are those for which $u = s$, and those edges are saturated, which means that they are not in the residual network.

Algorithm 1 GPR - INITIALIZE-PREFLOW(G,s)

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for each vertex v ∈ G.V
    v.h = 0
    v.e = 0
for each edge (u, v) ∈ G.E
    (u, v).f = 0
s.h = |G.V|
for each vertex v ∈ s.Adj
    (s, v).f = c(s, v)
    v.e = c(s, v)
    s.e = s.e − c(s, v)
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Algorithm 2 shows that the Initialization followed by a sequence of push and relabel operation, executed in no particular order gives the GENERIC-PUSH-RELABEL algorithm

Algorithm 2 GPR - GENERIC-PUSH-RELABEL(G,s)

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INITIALIZE-PREFLOW(G,s)
while there exists an applicable push or relabel operation
    select an applicable push or relabel operation and perform it
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1.1 Analysis

1. When Generic Push Relabel terminates, it produces a flow which is maximum

Lemma 1 *(An overflowing vertex can be pushed or relabelled)*

Let G=(V,E) be flow network with source s and sink t, let f be preflow and let h be any height function for f. If u is any overflowing vertex, then either a push or relabel applies to it

Proof: *For any edge (u,v), we have $h(u) \leq h(v)+1$, where h is the height function. But if the $h(u) \leq$ height of all the surrounding vertex then that is the condition for a relabel operation. This means that it is possible to always either push or relabel an overflowing vertex.* ■

Lemma 2 *(Height of a vertex increases by at least 1 after a relabel operation)*

Proof: *As stated in lemma 1, before a relabel operation $h(u) \leq h(v), \forall (u,v) \in E$. Then, after the relabel operation the height of u will 1 more than the minimum of the surrounding vertices i.e; $h(u) = 1 + min h(v)|(u,v) \in E$* ■

Lemma 3 *(Height constraints are never violated by push on relabel operations)*

Let G=(V,E) be a flow network with source s and sink t. Then the execution of the GENERIC-PUSH-RELABEL algorithm of G maintains the height function h.

Proof: *This proof is done by the induction of basic operations performed. If we take the PUSH operation on the edge (u,v) with heights $h(u) and h(v)$, we have $h(u) = h(v) + 1$ before the push. Then, after the PUSH we get two conditions of which for the non- saturating condition it creates 2 edges $(u,v) and (v,u)$ and so the height becomes $h(v) \leq h(u) + 1$. Then for the saturating push the edge (u,v) will not present after the push and arrives the same height $h(v) \leq h(u) + 1$. Noe for relabelling operation, it is divided into 2 cases: one where (u,v) is the edge and relabelling is happening at u and the second case where there is an incoming edge to u from w. For the first case before relabel, the height is given as $h(u) \leq h(v), \forall (u,v) \in E$ and after relabel height becomes $h(u) = 1 + min h(u) \leq h(v) + 1$, and this satisfies the height constraint for the edge (u,v). For case 2, this involves incoming edge and so using the inductive argument before relabel we have the height as $h(w) \leq h(u) + 1 \forall (w,u) \in E$ and after the relabel the height increase by 1 and as this was the condition before we arrive at $h(w) < h(u)+1$ by utilizing lemma 2. Thus, for both the cases the height constraint is satisfied and thus for both PUSH and RELABEL operations the height constraint is not violated.* ■

Lemma 4 *(If h is a height function satisfying the preflow f, then there are no $s-t$ paths in G)*

Let G=(V,E) be a flow network with source s and sink t, let f be a preflow in G, and let h be a height function on V. Then there exists no path from source s to sink t in the residual network G.

Proof: *This proof is done using contradiction. Assume that there exists a simple path p from source s to sink t, then let $p = (s = v_0, v_1, ..., t = v_k)$. Since p here is a simple path*

$k = |V| and (v_0, v_1, ..., v_k) \in E$. As h is the height function, $h(v_1) = h(v_2)+1, h(v_2) = h(v_3)+1$ and so on till $h(v_{k+1}) = h(v_k) + 1$. Now, combining all these inequalities over path p, we obtain $h(s) \leq h(t) + k$. Now as we have defined the height function, $h(s) = n$ and $h(t) = 0$ and since height is valid $\forall i \in (0, 1, 2, ..., k-1)$, such that $h(v_i) \leq h(v_{i+1}) + 1$. This results in $h(v_0) \leq h(v_1) + 1 \leq h(v_2) + 1 \leq ...... \leq h(v_k) + k \leq h(t) + k < n$, and this results in a contradiction. This means there cannot be any (s, t) path as long as the height function is valid. ■

Now from all the lemma's 1 to 4 are used to prove that whenever the GPR terminates the flow it produces is the maximum flow

A preflow where no vertex is overflowing is a flow. If f is a preflow, then after a PUSH (u, v), $f^{'}$ is also a preflow. The RELABEL after the PUSH operation also does not change the preflow because relabel only change the height. So, neither PUSH not RELABEL violates the preflow. From **lemma 1**, at termination we can say that no vertex is overflowing and so we have a flow at the termination. Now from **lemma 3**, we have a valid height function h at termination. But, now from **lemma 4** we found that if there is a valid height function h then there cannot be an (s, t) path in the G_f^*. So, by the maxflow-mincut theorem, the final preflow f^* is the maximum flow.

2. Generic Push Relabel terminates after $O(mn)$ RELABEL and $O(mn^2)$ PUSH operations

Lemma 5 *(If x is an overflowing vertex, then there is a $(x, s) path in G$)*

Let G=(V,E) be flow network with source s and sink t, and let f be a preflow in G. Then, for any overflowing vertex x, there is a simple path from x to s in the residual network G_f.

Proof: *Suppose U is the set of all vertices for which there is a path from x to s and we can take $U^{'}$ as the set of all the remaining vertices.*

Now if $s \notin U$

$\implies s \in V - U = U^{'}$

Because x is overflowing, $e(x) > 0$ and $x \in U$

$\implies \sum_{u\in U} \left(\sum_{v\in V} f(v, u) - \sum_{v\in V} f(u, v) \right) > 0$

Since $V = U \cup U^{'}$,

$\implies \sum_{u\in U} \left[\left(\sum_{v\in U} f(v, u) + \sum_{v\in U'} f(v, u) \right) - \left(\sum_{v\in U} f(u, v) - \sum_{v\in U'} f(u, v) \right) \right] > 0$

$\implies \sum_{u\in U} \sum_{v\in U'} f(v, u) - \sum_{u\in U} \sum_{v\in U'} f(u, v) > 0$

In other words we can say that there is a positive flow from U^i to U. If there is a flow, then there is an edge from U to $U^{'}$ and this means that there is an edge (u, v) where $u \in U$ and $v \in V$. This implies that v is reachable from x and it contradicts the first notion that v is not reachable from x. So, for every vertex x which is overflowing we can eventually push to s. ■

Lemma 6 *(The height of a vertex can never be more than $2n-1$)*

Let G=(V,E) be a flow network with source s and sink t. During the execution of the GENERIC-PUSH-RELABEL algorithm on G, we have $h \leq 2n-1$, (where $n = |V|$, the number of vertices) for all vertices $u \in V$

Proof: *The heights of source s and sink t never changes because these vertices are by definition not overflowing and so $h(s) = n$ and $h(t) = 0$. Now for some arbitrary vertex u before the relabel operation it is overflowing. Then, by **lemma 5** we can say that there exists a simple path from u to s and using the same argument used in **lemma 4** we can say that,* $h(u) \leq h(v_1) + 1 \leq ..... \leq h(s) + k \implies h(u) \leq 2n-1$ ■

Remark 1.1 *The bound on the number of relabel operations can be found by taking the number height increases per vertex time $(n-2)$. The height of the each vertex can be changed atmost $2n-)$ times, which gives the number of relabel operations as $O(n^2)$*

The pushes can be divided in saturating pushed and non-saturating pushes. A saturating push is when the total residual flow over the edge will be 0 meaning to saturate the edge capacity and disappears from the residual network. A non-saturating push is when we are pushing an excess amount of overflow from one vertex to another vertex, but the edge is not saturated, so the vertex from which the push happens will no longer have overflow after the non-saturating push.

Lemma 7 *(Bound on the number of saturating pushes is $O(mn)$)*

During the execution of GENERIC-PUSH-RELABEL on any flow network G=(V,E), the number of saturating pushes is less that $2mn$, where $m = |E|$ and $n = |V|$

Proof: *Here we can say that between two saturating pushes on (u, v), $h(v)$ increases by strictly greater than or equal to 2. Now before any pus, $h(u) = h(v) - 11$ and after a saturating push, that edge becomes saturated and is no longer in the residual network i.e; $(u, v) \notin E_f$. Before another (u, v) flow from v to u must increase and at some point there might have been a push from v to u. After the saturating push, $h(u) = h(v) + 1$ and so this means that $h(v)$ has increased by atleast 2.This implies that the bound on the number of saturating pushes is $O(mn)$.* ■

References and Further Reading

[1] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. 2009. Introduction to Algorithms, Third Edition (3rd. ed.). The MIT Press.