CS 5001 Homework - 1

Instructor Avah Banerjee

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Problem 1 Consider the following single-qubit gates:

$$U_1 = \frac{I + iX}{\sqrt{2}}, \quad U_2 = \frac{I + iY}{\sqrt{2}}, \quad U_3 = \frac{I + iZ}{\sqrt{2}}.$$

(a) For each of the unitaries U_1, U_2, and U_3, determine the axis of rotation on the Bloch sphere.

(b) Determine the rotation angle (in radians) for each gate.

(c) Explain briefly how you deduce the axis and angle from the matrix form (or equivalently, from knowledge of Pauli matrices and the Bloch sphere representation).

Hint: The Pauli matrices are

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 2 Consider the following function over three bits :

$$f(a, b, c) = (a \vee b) \oplus c$$

Construct a reversible version of the above function using additional bits (which are assigned to 0 initially). If f_r is the reversible function you have implemented determine the permutation that is represented by f_r. You do not have to reset the garbage to 0.

Problem 3 Consider the 2-qubit measurement that projects onto the four Bell states (also called the EPR pairs). The Bell states are:

$$
\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big), \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big), \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big), \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big).
\end{aligned}
$$

(a) Write down the projectors for each of the Bell states, i.e.

$$P_{\Phi^+} = |\Phi^+\rangle\langle\Phi^+|, \quad P_{\Phi^-} = |\Phi^-\rangle\langle\Phi^-|, \quad P_{\Psi^+} = |\Psi^+\rangle\langle\Psi^+|, \quad P_{\Psi^-} = |\Psi^-\rangle\langle\Psi^-|.$$

(b) Argue why these projectors form a complete measurement operator in the 2-qubit space.

(c) Give the spectral decomposition of the overall measurement operator (i.e., show that it can be described as a sum of rank-1 projectors with corresponding eigenvalues).

Problem 4 Determine what the following two qubit gate does (aka find the mapping $(x, y) \to (w, z)$):

$$A = |0\rangle\langle 1| \otimes I + |1\rangle\langle 0| \otimes X$$

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Problem 5 Let H be a Hermitian operator, i.e. $H^\dagger = H$. Consider the operator

$$U = e^{iH}.$$

(a) Show that $U^\dagger = e^{-iH}$.

(b) Prove that U is unitary, namely $UU^\dagger = U^\dagger U = I$.

(c) Give a short intuitive argument: why does a Hermitian operator generate a valid unitary when exponentiated by i?

Hint: You may use the power-series definition of the matrix exponential:

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$